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Permutation of permutations and Cayley's theorem

Open Mathematics Collaboration^{*†}

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Abstract

The idea behind this preliminary white paper is to try to understand the multiplication of the natural numbers as permutations. For this end, we construct a set of the permutation of permutations of the symmetric group S_3 .

keywords: symmetric group, permutations, abstract algebra

The most updated version of this white paper is available at

<https://osf.io/hd6ar/download>

Introduction

1. This white paper is waiting peer review and should therefore be treated as preliminary.
2. It is part of the global scholarly ecosystem published in the OJMP.
3. Online version
<https://bit.ly/3tjxcF7>

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Open Invitation

Review, add content, and co-author this white paper [6, 7].

Join the Open Mathematics Collaboration.

Send your contribution to `mplobo@uft.edu.br`.

Open Science

The **latex file** for this *white paper* together with other *supplementary files* are available in [8].

Ethical conduct of research

This original work was pre-registered under the OSF Preprints [9], please cite it accordingly [10]. This will ensure that researches are conducted with integrity and intellectual honesty at all times and by all means.

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+ Center for Open Science

<https://cos.io>

+ Open Science Framework

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Agreement

4. All authors agree with [7].

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1. Abstract

The idea behind this preliminary white paper is to try to understand the multiplication of the natural numbers as permutations. For this end, we construct a set of the permutation of permutations of the symmetric group S_3 .



2. Prerequisites



Function



One-to-one function (injection).



Onto function (surjection).



Ordered pair



Cartesian product



Binary operation



Bijjective function



Permutation



Homomorphism



Isomorphism



Function

Function from A to B

$$f : A \rightarrow B$$
$$\forall a \in A \exists! b \in B ((a, b) \in f)$$

$f, A, B :=$ sets

$\exists! :=$ exists exactly one

$(a, b) :=$ ordered pair

[1]



One-to-one function (injection)

$$f : A \rightarrow B$$

$$\neg \exists a_1 \in A \exists a_2 \in A (f(a_1) = f(a_2) \wedge a_1 \neq a_2)$$

[1]



Onto function (surjection)

$$f : A \rightarrow B$$

$$\forall b \in B \exists a \in A (f(a) = b)$$

[1]



Ordered pair

$$(a, b) = \{\{a\}, \{a, b\}\}$$

a := first coordinate

b := second coordinate

$[1, 2]$



Cartesian product

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

$A, B :=$ sets

$(a, b) :=$ ordered pair

[1]



Binary operation

$$\star : S \times S \rightarrow S$$

$S :=$ set

$S \times S :=$ Cartesian product

[2]



Bijjective function

Bijjective function := one-to-one + onto

[1]



Permutation

Permutation of A := bijection from A to itself

[3]



Homomorphism

f^h

$$f^h : G \rightarrow H$$

$$\forall x, y \in G : f^h(x * y) = f^h(x) \circ f^h(y)$$

$f^h :=$ function

$G, H :=$ sets

$\star, \circ :=$ binary operations

$(G, \star), (H, \circ) :=$ groups

[2, 4, 5]



Isomorphism

Isomorphism := bijective homomorphism

[2,4,5]



3. Group

$$(G, \star)$$

Associativity: $\forall x, y, z \in G, (x \star y) \star z = x \star (y \star z)$

Identity: $\exists e \in G : \forall x \in G, e \star x = x \star e = x$

Inverse: $\forall x \in G \exists y \in G : x \star y = y \star x = e$

$G :=$ set

$\star :=$ binary operation

[2]



4. Cayley's theorem

$$(G, \star) \cong (P, \circ_\alpha)$$

$(G, \star) :=$ group

$(P, \circ_\alpha) :=$ permutation group

\cong isomorphism

$\star, \circ_\alpha :=$ binary operations

$\alpha : P \rightarrow P$ (permutation $:=$ bijective function)

$\circ_\alpha :=$ composition of permutations

[2,3]



5. Permutation of permutations


Let $N_3 = \{1, 2, 3\}$. Suppose (N_3, \star) is a group.

$S_3 :=$ group of all the permutations of N_3

$$S_3 = \{(1), (12), (13), (23), (123), (132)\}$$

From **Cayley's theorem**, there is a bijection between (N_3, \star) and a **permutation group**.

 Brainstorming: constructing the set PP3

 Multiplication tables for S_2 and S_3



Brainstorming: constructing the set PP_3

$$S_3 = \{(1), (12), (13), (23), (123), (132)\}$$

Let $PP_3 = \{((1,1)), ((2,2)), ((3,3)), ((12,2)), ((13,3)), ((23,6)), ((123,6)), ((132,6))\}$.

$((1,1))$ is the permutation of (1) by (1).

$((2,2))$ is the permutation of (2) by (2).

$((3,3))$ is the permutation of (3) by (3).

$((12,2))$ is the permutation of (12) by (2).

$((13,3))$ is the permutation of (13) by (3).

$((23,6))$ is the permutation of (23) by (6).

$((123,6))$ is the permutation of (123) by (6).

$((132,6))$ is the permutation of (132) by (6).

Clearly there is a bijection between S_3 and PP_3

.



Multiplication tables for S_2 and S_3

Check [11] at <https://doi.org/10.31219/osf.io/r3jvu> or here.

$S_2 = \{(1), (12)\}$, let $a := (1)$ and $b := (12)$

S_2	a	b
a	a	b
b	b	a

$S_3 = \{(1), (12), (13), (23), (123), (132)\}$

$a := (1)$, $b := (12)$, $c := (13)$, $d := (23)$, $e := (123)$, $f := (132)$

S_3	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	a	e	f	c	d
c	c	f	a	e	d	b
d	d	e	f	a	b	c
e	e	d	b	c	f	a
f	f	c	d	b	a	e



6. Final Remarks

We present a bijection from PP_3 (the permutation of permutations of N_3) and a group (\mathbb{N}_3, \star) in order to have an insight of (\mathbb{N}_3, \star) as a permutation group, a known result from Cayley's theorem.



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